University of Tabriz

Journal of Civil and Environmental Engineering
Volume 51 (2021), Issue 2 (Summer), 21-28

EXTENDED ABSTRACT

Developing a New Semi- Analytical Method for Solving Elastodynamic Problems in the Frequency Domain

Reza Babaee [a], Ehsan Jabbari [a,*], Mohsen Khosravi Babadi [b]

[a] *Faculty of Engineering, University of Qom, Qom, Iran*
[b] *Iran Marine Industries Organization*

Received: 27 March 2017; **Accepted:** 15 July 2019

Keywords:
Elastodynamic problems, Diagonal coefficient matrix, concrete gravity dam, Frequency domain.

1. Introduction

In this paper, a new semi-analytical method is developed for analyzing concrete gravity dams in the frequency domain. Among different numerical methods, the finite element method (FEM), the boundary element method (BEM), and the scaled boundary finite element method (SBFEM) are more popular. BEM requires basically reduced surface discretization and may be considered as an appealing alternative to FEM for elastodynamic problems but requires fundamental solution of the governing differential equations. Although coefficient matrices of BEM are much smaller than those of FEM, they are routinely non-positive definite, non-symmetric, and fully populated. The SBFEM combines the advantages of the FEM and the BEM. The SBFEM is a semi-analytical method for solving partial differential equations by transforming the governing partial differential equations to ordinary differential equations. In the SBFEM, similar to the BEM, the boundary of the problem's domain is discretized, while no fundamental solution is required. A modified form of the SBFEM with diagonal coefficient matrices has been proposed (Fakharian amd Khodakarami, 2015) for solving elastodynamic problems in the time domain. In this study, the semi-analytical approach for solving elastodynamic problems in the frequency domain has been applied, the governing equations in local coordinate system has been developed and two concrete gravity dams with rigid foundations and empty reservoir have been analyzed under the earthquake harmonic load.

2. Methodology

2.1. Governing equations in global coordinates

The equation of motion for elastodynamic problems under earthquake load in a 2D domain is represented as:

$$\sigma_{ij,j} - \rho(\ddot{u}_i + \ddot{u}_g) = 0 \tag{1}$$

Where σ_{ij} shows the stress tensor components, $\ddot{u}_g$ refers to ground acceleration, $\ddot{u}_i$ refers to relative acceleration of the structure and ρ is the mass density. For a 2D domain in global Cartesian coordinates, $i = X, Y$ and $j = X, Y$. In the frequency domain, the time derivative of the displacement function, $\ddot{u}_i(t)$, may be given as:

$$u_i(t) = \hat{u}_i(\omega) exp(I\omega t) \tag{2}$$

in which $I = \sqrt{-1}$ and $\hat{u}_i(\omega)$ indicates the displacement amplitude. Therefore, governing equations in frequency domain is formulated as:

* Corresponding Author
E-mail addresses: r.babaee@stu.qom.ac.ir (Reza Babaee), e.jabbari@qom.ac.ir; (Ehsan Jabbari), mohsen_khb@yahoo.com (Mohsen Khosravi Babadi).

$$\hat{\sigma}_{ij,j} + \rho\omega^2\hat{u}_i + \rho\omega^2\hat{u}_g = 0 \tag{3}$$

Where ω denotes the circular frequency. The present method uses the weak form of the governing equations. For this end, Eq. (3) is weighted with an arbitrary weighting function w and integrated over the problem's domain along with applying appropriate BCs. The result may be given by:

$$\int_{\Omega} w\hat{\sigma}_{ij,j}\, d\Omega + \int_{\Omega} w\omega^2\hat{u}_i d\Omega + \int_{\Omega} w\rho\omega^2\hat{u}_g\, d\Omega = 0 \tag{4}$$

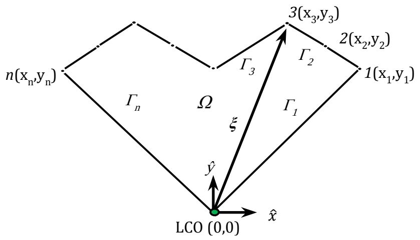

Fig. 1. Geometry of a sample 2D bounded domain (Ω) and LCO location in global coordinates

2.2. Geometry modeling

In the present method, for a bounded medium, a coordinates-origin (LCO) is chosen from which all boundaries of the domain are visible (Fig. 2). A geometry transmission is introduced from global Cartesian coordinates (x, y) to local dimensionless coordinates (ξ, η) (Fig. 2). This transmission is obtained by Lagrange polynomials as mapping functions (Canuto et al, 2012) as:

$$\varphi_i(\eta) = \prod_{k=1,k\neq i}^{n_\eta+1} \frac{\eta - \eta_k}{\eta_i - \eta_k} \tag{5}$$

2.3. Physical modeling

In this method, special polynomials $N(\eta)$ are used as shape functions. Diagonal coefficient matrices will be derived by using these shape functions. To this end, the displacement function and its derivatives, across the element, are interpolated using polynomials that own two specific characteristics; the shape functions have Kronecker Delta property, and their first derivatives are equal to zero at any given control point.

For an element by (n_η+1) nodes, the shape functions are expressed as a polynomial of degree ($2n_\eta$+1) as (Babaee et al., 2015):

$$N_i(\eta) = \sum_{m=0}^{2n_\eta+1} a_m \eta^m \tag{6}$$

Considering Eqs. (2) and (6), the displacement field at any point (ξ, η) and frequency ω is given by

$$\{\hat{u}(\xi,\eta,\omega)\} = [N(\eta)]\{\hat{u}(\xi,\omega)\} = [N(\eta)]\left[\hat{u}_x(\xi,\omega) \quad \hat{u}_y(\xi,\omega)\right]^T \tag{7}$$

2.4. Numerical integration

In this study, the Gauss-Lobatto-Legendre numerical integration method is applied. This method calculates the values of the coefficients matrix and vector that will be appeared in governing equations in local coordinates, according to the node element that corresponds to the points and also features a shape function used, resulting diagonal matrix of coefficients used in the equation. Weight coefficients used in the method of integration is calculated using (Canuto et a, 2012):

$$w_i = \frac{2}{n(n+1)[(n_\eta)(n_i)]^2} \tag{8}$$

3. Results and discussion

3.1. Derivation of governing equations in local coordinates

The weak form of governing equations (Eq. 4) is derived as Eq. 9 using mapping function (Eq. 5), shape function (Eq. 6), and numerical integration (Eq. 8).

$$\xi\begin{bmatrix} D^0_{11x} & 0 & \cdots & 0 & 0 \\ 0 & D^0_{11y} & & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & & D^0_{nnx} & 0 \\ 0 & 0 & \cdots & 0 & D^0_{nny} \end{bmatrix}\begin{Bmatrix} \hat{u}_{1x} \\ \hat{u}_{1y} \\ \vdots \\ \hat{u}_{nx} \\ \hat{u}_{ny} \end{Bmatrix}_{,\xi\xi} + \begin{bmatrix} D^1_{11x} & 0 & \cdots & 0 & 0 \\ 0 & D^1_{11y} & & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & & D^1_{nnx} & 0 \\ 0 & 0 & \cdots & 0 & D^1_{nny} \end{bmatrix}\begin{Bmatrix} \hat{u}_{1x} \\ \hat{u}_{1y} \\ \vdots \\ \hat{u}_{nx} \\ \hat{u}_{ny} \end{Bmatrix}_{,\xi} + \omega^2\xi\begin{bmatrix} M_{11x} & 0 & \cdots & 0 & 0 \\ 0 & M_{11y} & & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & & M_{nnx} & 0 \\ 0 & 0 & \cdots & 0 & M_{nny} \end{bmatrix}\begin{Bmatrix} \hat{u}_{1x} \\ \hat{u}_{1y} \\ \vdots \\ \hat{u}_{nx} \\ \hat{u}_{ny} \end{Bmatrix} + \xi\begin{Bmatrix} \hat{F}^b_{1x} \\ \hat{F}^b_{1y} \\ \vdots \\ \hat{F}^b_{nx} \\ \hat{F}^b_{ny} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{Bmatrix} \quad (9)$$

Where:

$$D^0_{ij} = 2\delta_{ij} w_i [B^1(\eta_i)]^T [D][B^1(\eta_i)] |J(\eta_i)| \quad (10)$$

$$D^1_{ij} = 2\delta_{ij} w_i [B^1(\eta_i)]^T_{,\eta} [D][B^2(\eta_i)] |J(\eta_i)| \quad (11)$$

$$M_{ij} = 2\delta_{ij} w_i [N(\eta_i)]^T \rho [N(\eta_i)] |J(\eta_i)| \quad (12)$$

$$\hat{F}_i{}^b(\omega)_i = 2\delta_{ij} w_i [N(\eta_i)]^T \rho\omega^2 \{\hat{u}_g\} |J(\eta_i)| \quad (13)$$

and $\hat{F}^b(\omega) = [\hat{F}^b_x(\omega) \quad \hat{F}^b_y(\omega)]^T$ are the components of inertial forces caused by foundation excitation in the domain of dam body at a frequency ω and δ_{ij} denotes the Kronecker Delta which results in diagonal coefficient matrices. For calculating deformations and stresses at every Degree Of Freedom (DOF), the differential equation corresponding to the control point related to the DOF should be solved. Analytical solution for governing equation for each DOF may be represented as:

$$\hat{u}_{ci} = A_i \xi^{\left(\frac{1-\frac{D^1_{ii}}{D^0_{ii}}}{2}\right)} J_{\left(\frac{\frac{D^1_{ii}}{D^0_{ii}}-1}{2}\right)}\left(\left(\frac{\omega^2 M_{ii}}{D^0_{ii}}\right)^{0.5}\xi\right) + B_i \xi^{\left(\frac{1-\frac{D^1_{ii}}{D^0_{ii}}}{2}\right)} Y_{\left(\frac{\frac{D^1_{ii}}{D^0_{ii}}-1}{2}\right)}\left(\left(\frac{\omega^2 M_{ii}}{D^0_{ii}}\right)^{0.5}\xi\right) - \frac{1}{\omega^2 M_{ii}}\hat{F}^b_i \quad (14)$$

in which $J(\alpha)$ (β) and $Y(\alpha)$ (β) indicate respectively the first and second kinds of Bessel functions of order α.

3.2. Numerical examples

The accuracy of the present method is demonstrated through representative numerical examples. Geometry of a concrete gravity dam with rigid foundation duo to harmonic horizontal displacement of ground is shown in Fig. 2 and results of the analysis using the present method is shown in Fig. 3.

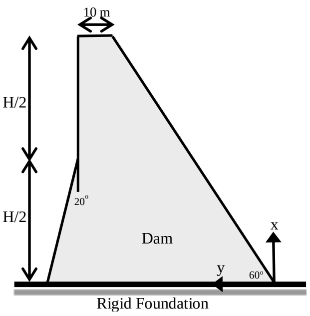

Fig. 2. Geometry of a concrete gravity dam

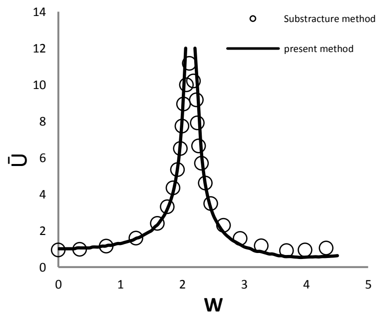

Fig. 3. Amplitude variations of horizontal displacement of dam crest

4. Conclusion

In this research, a new semi-analytical method with detailed formulation was presented for the analysis of 2D elastodynamic problems in frequency domain. In this method, only the boundaries of the domains are discretized. Using Lagrange polynomials as mapping function, special shape function, Gauss- Lobatto- Legendre quadrature, and implementing a weak form of weighted residual method, coefficient matrices of the system equations become diagonal. Therefore, the partial differential equation for each DOF becomes independent from others. Consequently, this method significantly reduces the computational costs compared to other methods. Besides, two examples of empty gravity dam were successfully modeled with very small number of DOFs, preserving very high accuracy compared to available solutions.

5. References

Babaee R, Khaji N, Ahmadi MT, "Development of Decoupled Equations Method for calculating hydrodynamic pressures on concrete gravity dams", Modarres civil engineering journal, 2015, 15, 41-52.

Fakharian M, Khodakarami MI, "A Modified Decoupled Semi- Analytical Approach Based On SBFEM for Solving 2D Elastodynamic Problems", World Academy of Science, Engineering and Technology, International Journal of Mechanical, Aerospace, Industrial, Mechatronic and Manufacturing Engineering, 2015, 9 (2), 296-301.